An experimental investigation of turbulent spots and breakdown to turbulence

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The theory of hydrodynamic stability and the impact on it of recent work with turbulent spots is discussed. Emmons's (1951) assumptions about the growth and interaction of turbulent spots are found experimentally to be substantially correct. In particular it is shown that the region of turbulent flow on a flat plate is simply the sum of the areas that would be obtained if all spots grew independently.

An investigation of the conditions required for breakdown to turbulence near a wall, that is, to initiate a turbulent spot, suggests that regardless of how disturbances are generated in a laminar boundary layer and independent of both the Reynolds number and the spatial extent of the disturbances, breakdown to turbulence occurs by the initiation of a turbulent spot at all points at which the velocity fluctuation exceeds a critical intensity. Over most of the layer this intensity is about 0·2 times the free-stream velocity. The Reynolds number is important merely in respect of the growth of disturbances prior to breakdown.

1. Introduction

Ever since Reynolds classical experiments, considerable attention has been paid to the problem of the origin of turbulence. Nevertheless, it is only in recent years that a partial understanding of the physical processes involved has been achieved. Throughout this paper we consider only the transition to turbulence on a semi-infinite flat plate lying in the Cartesian plane $y = 0$ with its leading edge along $x = 0$ in a uniform stream of velocity U_1 in the x-direction. Even under the most carefully controlled conditions the free stream will also contain small velocity fluctuations of order $0{\cdot}001\,U_1$, and indeed it is an immediate consequence of the linear stability theory that without such fluctuations the layer which grows on the plate will remain laminar. In other words a laminar boundary layer cannot spontaneously, without the further action of an external disturbance, become turbulent.

The hydrodynamic stability approach

The early work considered only the initial growth of infinitesimal disturbances superimposed on the laminar undisturbed flow (Schlichting 1955). In 1948 Schubauer & Skramstad brilliantly confirmed the existence of the linear waves postulated by this linear theory of the instability of the boundary layer on a flat plate. A more exact verification of the details of the theory has been given by Shen

(1954). When a small disturbance is applied to a laminar flow for which the Reynolds number exceeds a critical value, initially the disturbance grows exponentially with time. While the disturbance is small the Reynolds stress generated by the disturbance is negligibly small, but as the disturbance amplitude increases the finite velocity fluctuations transport appreciable momentum, and the associated Reynolds stress modifies the mean flow so that the transport of energy from the mean flow to the disturbance is modified and the disturbance growth rate is affected. Recently attempts have been made, notably by Stuart (1958), to extend the wave instability theory to the non-linear case of finite disturbances, but so far only flows for which the local Reynolds number is independent of position in the flow have been considered. Nevertheless, in spite of the success of this work on the growth of two-dimensional waves of infinite streamwise extent, the stability theory contains no feature typical of turbulence and can at best describe the early stages of the flow before the onset of turbulence. In fact, while the suggestion (Landau 1944) that a complex non-linear superposition of two-dimensional waves of various wavelengths and various wave directions is turbulence is plausible for, say, convection between parallel planes, this is certainly not the case for turbulence on a flat plate which originates in very small regions as 'turbulence spots' (Emmons 1951). While turbulent spots can be initiated from high-amplitude portions of a superposition of boundary-layer waves, they can also be initiated by isolated velocity pulses produced say by a spark.

Further, the linear theory shows that below a critical Reynolds number all waves are damped, and this has been verified in detail by experiment. Consideration of the non-linear equations for a flow with varying Reynolds number (Stuart 1956, 1958) has not yet produced a definite conclusion as to whether or not a finite two-dimensional wave is also damped below the critical Reynolds number. Nevertheless, it will be shown below that for a wide class of finite disturbances it is possible to initiate turbulent spots at Reynolds numbers well below the critical value (provided the amplitude exceeds a certain value). In other words, below the critical Reynolds number the flow is stable to small disturbances but unstable to large disturbances. This is a common stability situation. For example, consider a ball resting inside a bowl. After a small disturbance the ball performs a damped oscillation and returns to its original position, but a sufficiently large impulse will make the ball reach the rim and leave the bowl. It is clear that a desirable next step would be a theoretical consideration of the growth in a laminar boundary layer of an isolated velocity pulse.

Emmons theory of a transition zone

A second effect of the modification of the rate of transfer of energy to a disturbance by the Reynolds stresses is the extremely rapid increase in disturbance energy which initiates a turbulent spot. In 1951 Emmons reported the existence of turbulent spots and gave a probability calculation of their interaction in a transition zone. Subsequent work by Dhawan & Narasimha (1958) has established a remarkably good phenomenological theory of a transition zone.

Emmons's theory relies on four assumptions: (1) point-like breakdown, (2) a sharp boundary between the turbulent fluid of a spot and the surrounding laminar flow, (3) a uniform rate of spot growth and (4) no interaction between spots. These ideas were experimentally investigated in detail by Schubauer & Klebanoff (1955). In practice point-like breakdown means that the turbulent spots originate from a region of extent less than the boundary-layer thickness. If a hot-wire anemometer is placed close to the beginning of a transition zone, the observed turbulent spots will be found to be of small extent. Further, under suitable conditions, an artificial disturbance will initiate a turbulent spot even though the disturbance is of small extent. This can be very clearly demonstrated by the dye method described below in connexion with figure 2 (plate 1). (Nevertheless, the smallest volume of disturbed fluid which will initiate a turbulent spot is not yet known.) Schubauer & Klebanoff (1955) have also verified the assumptions of a sharp boundary and uniform growth but did not give detailed consideration to the interaction assumption, although they indicated that they made qualitative observations in an experiment similar to that below. It will be one of the objects of this paper to give, in §3, experimental evidence for Emmons's fourth assumption that the area of a flat plate covered with turbulent fluid is simply the combined area of the turbulent spots present assuming each spot grows independently of all the others. Thus, in view of the detailed validity of Emmons's assumptions, it is not surprising that the consequent theory of a transition zone is in excellent agreement with experiment.

Breakdown

We have considered the fluid behaviour subsequent to breakdown, but the word breakdown correctly implies that the spots originate from a very small volume and suggests the importance of the disturbance present immediately before breakdown occurs. It is important to note that this does not imply that the character of the flow far upstream of the breakdown point is unimportant, but rather that the role of the upstream flow is to produce and amplify disturbances of which those that satisfy certain local conditions will break down to turbulence. Indeed, the conclusion that breakdown is determined by local conditions is the principal result of this paper.

It has already been noted that spots can be initiated by an isolated velocity pulse established in the laminar boundary layer. Although to specify a pulse in general it would be necessary to give the velocity fluctuation as a function of position and time, an approximate specification is obtained from the longitudinal extent (from zero to zero) in the flow direction X, the lateral extent Z, and the maximum value of the velocity fluctuation in the flow direction $u_m(y)$ as the pulse passes over the observation point. This specification ignores the other velocity components and the fluid acceleration, but it will appear below that a more detailed specification within the experimental accuracy is not required. Let such a pulse pass over a point at (x, y), where the undisturbed velocity profile is $U(y)$ and the boundary-layer thickness is δ ($U = 0{\cdot}99U_1$ at $y = \delta$). The subsequent behaviour of the pulse is determined by the seven quantities x, y, U_1, or $U(y)$, X, Z, u_m, and either kinematic viscosity ν or δ ($= 5\sqrt{(\nu x/U_1)}$); from these are obtained the convenient dimensionless parameters the Reynolds number $R = U_1 x/\nu$, y/δ, u_m/U, X/δ and Z/δ. The purpose of the experiment described in §4 is to investigate the relation between these five parameters for pulses on the verge of breakdown. It will be shown that breakdown occurs for all pulses for which $u_m/U \geqslant u^*$, where u^* is a function only of y/δ whatever the Reynolds number or the spatial extent of the pulse. This surprisingly simple result confirms the suggestion that breakdown is determined only by local conditions.

Considerable evidence has already been presented by Klebanoff & Tidstrom (1959) to support the result that breakdown depends critically on u^*. They produced a boundary-layer wave downstream of a vibrating ribbon and found that breakdown was first noticed in the high amplitude portions of the wave. From measurements restricted to $y/\delta \doteqdot 0{\cdot}2$ and over a small Reynolds number range, they found that this occurred when $u'/U = 0{\cdot}22$, where u' is the root-mean-square x-velocity fluctuation. Unfortunately, from Klebanoff & Tidstrom's data it is difficult to relate u_m and u', but just before breakdown was noticed the wave shape was composed of long rather rectangular pulses so that $0{\cdot}7 < u_m/u' < 1$ and $u^* \doteqdot 0{\cdot}25$. That this is somewhat in excess of the value 0·18 to be given here is to be expected. While Klebanoff & Tidstrom find the value of u'/U at which breakdown is first observed at $y/\delta = 0{\cdot}2$ it is clear, as they point out, that this is not necessarily the point at which breakdown first occurs. In fact from their data we find the maximum value of u'/U just prior to breakdown is near $y/\delta = 0{\cdot}3$. Near breakdown there is a very rapid increase in u'/U to a maximum value about twice the breakdown value, so that a small error in locating the position at which breakdown first occurs at all in the layer leads to a large overestimate of the critical value of u'/U. Nevertheless, considering that the possible systematic error in determining u/U or u'/U by means of a hot-wire is rather large (10–20 %), it is doubtful if there is any significant difference between Klebanoff & Tidstrom's estimate of u^* and the values given here. Finally, it should be noted that although in the preliminary investigation of this problem a number of different hot-wires were used, all the data presented here was obtained with a single hot-wire which maintained its calibration throughout the experiment, so that although the absolute value of u^* may be somewhat in doubt it was clearly shown that breakdown occurred at a critical value of u^*.

2. The role of the Reynolds number

Behaviour of a disturbance generator

It has been suggested that the Reynolds number is important only in the growth of disturbances prior to breakdown, and in fact it is the dominance of the Reynolds number in this growth that is responsible for the belief that the Reynolds number is also important for breakdown. Those investigations, such as that of Webb & Harrington (1956), which report the Reynolds number of transition as a function of the amplitude of a disturbance generator refer to the operation of the generator rather than to the origin of turbulence. Such investigations do, however, make clear the obvious but important point that at low Reynolds numbers very much larger amplitudes of the disturbance generator are required to produce a given u/U. In fact any such device is completely inhibited for U_1 sufficiently small.

Consider in detail a particular case (Schlichting 1959), that of transition behind a circular cylinder of diameter k laid normal to the stream at x on a flat plate where the undisturbed boundary-layer thickness is δ. The location of the beginning of the transition zone x_t will be a function of k, x, δ or ν, U_1 or $U(k)$, and some measure of the free-stream disturbances such as u' the root-mean-square x-velocity fluctuation. Suitable dimensionless parameters are $R_t = U_1 x_t/\nu$, $R_k = U(k)\,k/\nu$, k/δ or x/k, u'/U_1. The experimental data, given by Dryden (1953), is reproduced

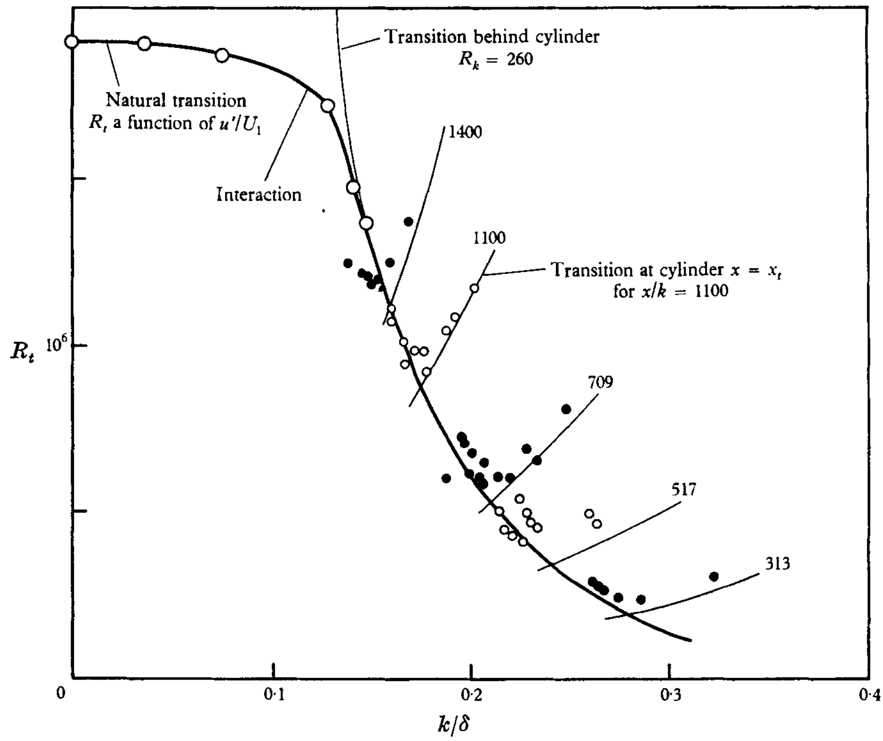

FIGURE 1. Reynolds number of transition behind a cylinder of diameter k placed normal to the stream on a flat plate. Data given by Dryden (1953). Below the heavy curve the flow cannot be turbulent. ◯, k varied; ○, ●, δ varied.

in figure 1 as a relation between R_t and k/δ. Low values of k/δ were obtained by varying k with fixed x, δ, but the higher values of k/δ were obtained with fixed x/k by varying δ. For k/δ very small the flow is unaffected by the presence of the roughness element. Here $R_k < 10$ so that the boundary layer does not separate from the cylinder, no vortices are shed and R_t is determined by u'/U_1 as in natural transition. Schubauer & Skramstad (1948) have investigated this relation and found that R_t decreases as u'/U_1 increases. It is worth noting that in most wind tunnels u'/U_1 is a function of $U_1 M/\nu$, where M is a length characteristic of the upstream section of the tunnel (say the gauze spacing). The relation between R_t and k/δ for $k/\delta < 0{\cdot}15$ is consistent with the result of the present paper and is expressed implicitly by $u_m/U = u^*$, where u_m is the maximum value of u at a given point. The result of the present paper is not applicable in any other region of the flow problem under discussion.

As k increases beyond $R_k \doteqdot 100$ the boundary layer separates at the cylinder and vortices are shed. These vortices interact with the free-stream disturbances, so that the transition point is moved towards the cylinder until for k sufficiently large transition occurs in the shear layer of the separation zone. Transition in the separated shear layer will be determined by $U(k)$, k, ν; that is by R_k. The curve $R_k = 260$ gives the best fit of Dryden's data. This value is reasonable but a little lower than the value $R_k = 300$ at which visual inspection of the flow in a water channel showed the first signs of disturbances in the separated shear layer. Further increase in k cannot move the transition point past the cylinder position x. This is seen in figure 1 where the points taken at constant x/k follow the curves $R_t = U_1 x/\nu$ for larger values of k/δ. The points will begin to approach the curves $R_t = U_1 x/\nu$ when transition first occurs in the separated shear layer.

Natural transition

In natural transition on a flat plate six zones can be distinguished in practice. First, there is a laminar zone in which the disturbance amplitude is less than that in the free stream. Secondly, an oscillatory zone commences at

$$R = R_c = 1{\cdot}12 \times 10^5.$$

In this zone the fluctuating motion is largely a rapidly growing two-dimensional boundary-layer wave. With care this zone can be made to extend at least to $R = 3 \times 10^6$ (Schubauer & Skramsted 1948). The third zone is entered when the boundary-layer wave begins to interact with the free-stream disturbances. It can be shown that a vortex line moving near the wall in a boundary layer is unstable to small disturbances along its length. Even the most careful attempt to reduce the free-stream disturbances does not prevent the wave front becoming rapidly distorted (Hama, Long & Hegarty 1956). The behaviour of the first three zones is dominated by the Reynolds number. Generally by this stage the disturbance intensity is approaching u^* and the fluid passes into a fourth zone in which breakdown occurs at the high intensity points of the wave front where the intensity reaches u^*. This breakdown zone is narrow (Dhawan & Narasimha 1958) because the rate of growth of the velocity fluctuations is rapid in the third zone. In the fifth zone, the so-called transition zone, the spots grow linearly—evidence that they are only weakly affected by the Reynolds number—until they merge into the sixth zone of full turbulence which is again dominated by the Reynolds number. It is perhaps not entirely surprising that the two regions, laminar and turbulent, each differently dependent on the Reynolds number should be separated by a breakdown region independent of the Reynolds number.

3. Turbulent spots, experimental results

Form and growth

Turbulent spots can be conveniently visualized by placing a flat plate in a water channel and injecting dye through a slit parallel to the plate leading edge, or more simply by sprinkling dye particles onto the plate so that dye spreads downstream in a thin layer which remains close to the plate wall. If now a localized

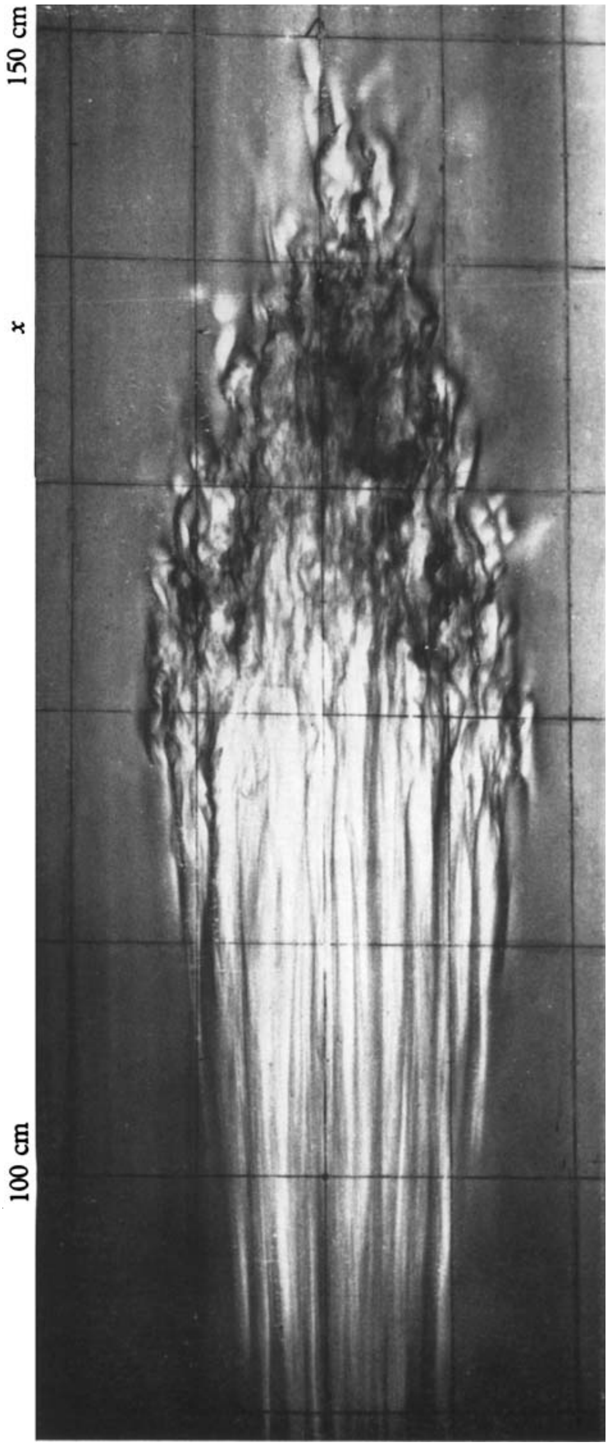

FIGURE 2. Photograph of a turbulent spot 3·9 sec after a 0·1 sec disturbance at $x = 50$ cm on a flat plate. $U_1 = 23$ cm/sec, $\delta = 1\cdot2$ cm at $x = 130$ cm.

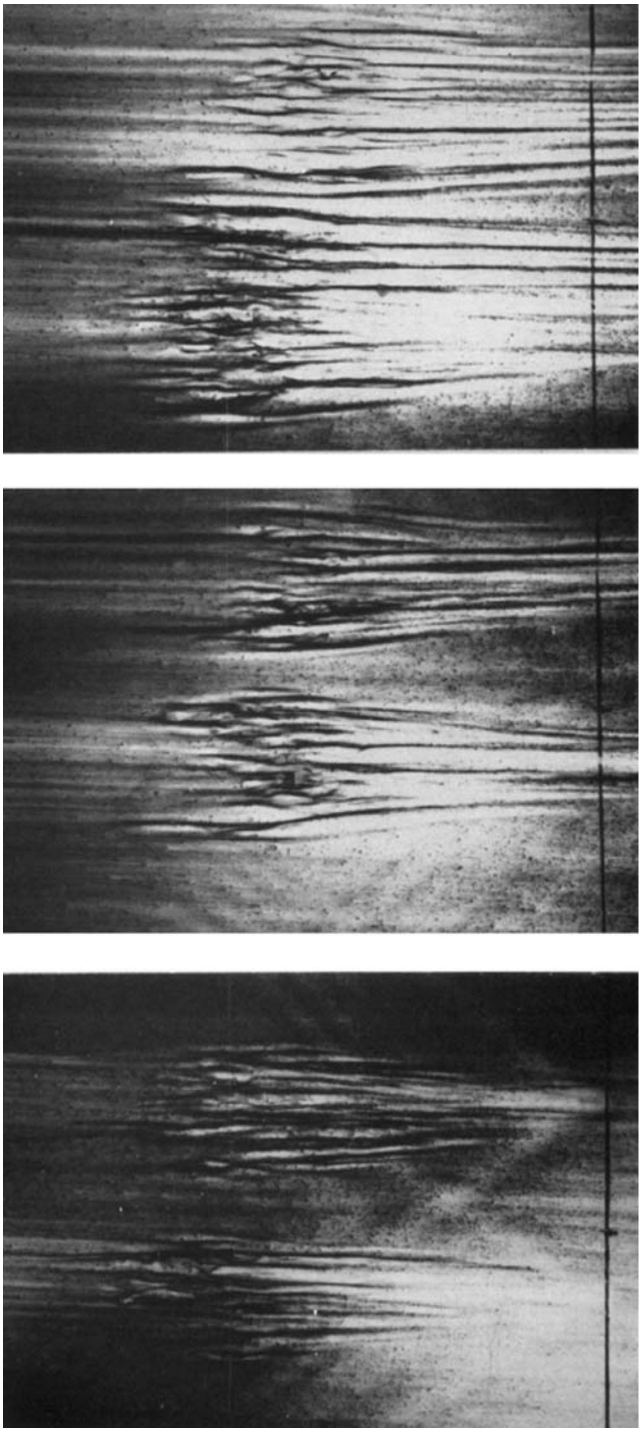

FIGURE 4. Photographs of two spots initiated 5 cm apart in a classroom demonstration channel. Water depth 5 mm, discharge velocity 32 cm/sec. The lines are spaced 20 cm apart. The photographs show an area approximately 15 × 10 cm.

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disturbance is produced at some point, say the end of a rod momentarily poked into the boundary layer, the motion produced by the disturbance as it moves downstream is clearly revealed by breaks or clear patches occurring in the dye sheet. If the disturbance is large enough a turbulent spot will be initiated. Figure 2, plate 1, shows a photograph of the disturbed dye sheet produced by a turbulent spot, obtained by this method in the 14 in. water flume of the Engineering Laboratory, Cambridge. The photograph was taken 3·9 sec after a disturbance lasting 0·1 sec was produced 50 cm downstream of the plate leading edge with a free stream of velocity $U_1 = 23{\cdot}0$ cm/sec and $\delta = 1{\cdot}2$ cm at $x = 130$ cm.

The general features of a spot as revealed by figure 2 (plate 1), have already been obtained from hot-wire measurements by Schaubauer & Klebanoff (1955). To the right of the photograph is a V-shaped region of turbulent fluid lying 115–145 cm downstream of the leading edge and to the left the fluid returns to laminar flow as the spot passes downstream to the right. The striations in this laminar region are due to elements of dye which are rapidly extended in the flow direction by the high shear near the plate. The turbulent–laminar boundary is sharply defined, although distorted on a scale comparable with the boundary-layer thickness. This property of a sharp division between laminar and turbulent fluid appears to be one of the general characteristics of turbulent fluid. Within the spot itself a large portion of the fluctuating motion is surprisingly regular and periodic. The principal components of the fluctuating motion appear very similar to so-called horseshoe vortices; that is, regions of localized vorticity, the core of which is U-shaped and parallel to the plate but with the head of the U pointing downstream and moving away from the wall. They appear similar to structures identified by Grant (1958) in a turbulent boundary layer. As yet, however, no detailed investigation of the structure of spots has been reported.

Interaction

It was mentioned that little attention has been given to Emmons's assumption which relates to the manner in which turbulent spots interact. This problem of the interaction of spots is also of interest in connexion with transition on a finite flat plate (to be published). It has been shown experimentally that transition occurs near the side edges of a finite plate much sooner than transition in the middle of the plate. Turbulent spots originate from a small volume on the edge of the plate and as they grow downstream sweep out a narrow tongue of disturbed fluid near the edge, while the bulk of the flow over the plate is still laminar. We then have the problem of the manner in which the edge regions combine with the normal transition zone, or if the plate is very narrow with one another. Both these problems are most simply investigated by studying the interaction of two equal but laterally displaced spots.

A thin aluminium flat plate was set parallel to the incident stream ($U_1 = 820$ cm/sec) in the 15 in. wind tunnel of the Cavendish Laboratory, Cambridge. Sparks were fired simultaneously at a rate of 2/sec across two 3 mm spark gaps placed 25 cm downstream of the leading edge and 5 cm apart so that two spots were simultaneously initiated 5 cm apart at the spark gaps. The subsequent interaction downstream of the spark gaps was studied with a hot-wire placed so that $y/\delta = 0{\cdot}4$

and connected to a triggered oscilloscope, the triggering signal for which was obtained from the spark generator. The proportion of time γ that the flow at a point was turbulent was obtained directly from the pulse length observed on the oscilloscope. The results are shown in figure 3 as a series of curves of constant γ. (Notice that here γ is proportional to the pulse rate.) Superimposed on the figure are two overlapping shaded areas representing the outline of each of the turbulent spots at a particular instant assuming each spot grows independently of the other. (The outline was traced from that of figure 2, plate 1.) Figure 3 shows that the assumption is correct. An isolated spot grows linearly in time as is verified here

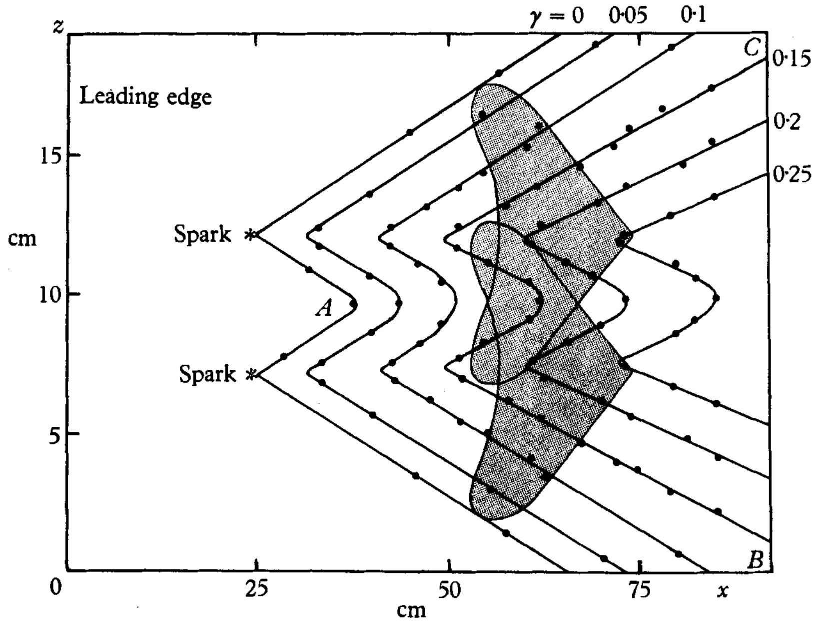

FIGURE 3. Interaction of two equal spots initiated 5 cm apart at $x = 25$ cm. Curves of constant intermittency γ for a fixed repetition rate of 2 pulses/sec. The overlapping shaded areas represent the outline of the two spots at a particular instant. The outline was copied from figure 2 (plate 1). Scale 1:4.

where the lines of constant δ are largely made up of straight lines. But this is also the case within the possible interaction zone, the triangular area BAC. A small interaction zone does exist on the line of symmetry through A, but its width is only of order δ as might be expected, since the edge of a spot is contorted with a scale of order δ. A simple demonstration of the independent growth rate was given by noting that γ was unaltered in BAC (except for the narrow strip of width δ through A) if either spark was switched off and also unaltered below AB if the upper spark was switched off. Thus the curve dividing turbulent from laminar fluid is the envelope of the superimposed curves from independent individual spots.

The interaction of spots can also be studied qualitatively in the same manner as that used to obtain figure 2 (plate 1). Figure 4 (plate 2), shows a series of photographs of two spots initiated 5 cm apart in a shallow flume used for classroom demonstrations. The flume was 30 cm wide and water was supplied up to

10 gal./min from a laboratory tap. The difference in the shape of the spots in figures 2 and 4 is due to the fact that in figure 4 the undisturbed flow is independent of x. A visual study of the interaction of spots in the water flume confirmed that there was no noticeable alteration in the growth rate of one spot due to the presence of another.

4. Turbulent breakdown, experimental results

We wish to know the conditions under which an isolated and localized pulse will initiate a turbulent spot. The main difficulty in determining these conditions accurately is in producing a regular sequence of pulses as near similar as possible. The first method tried was that of a small circular cylinder momentarily introduced into the boundary layer through a hole in the plate. (The alternatives of a cylinder introduced from the free stream or a puff of air through a hole in the plate were not tried.) The cylinder was mounted on the armature of an electric

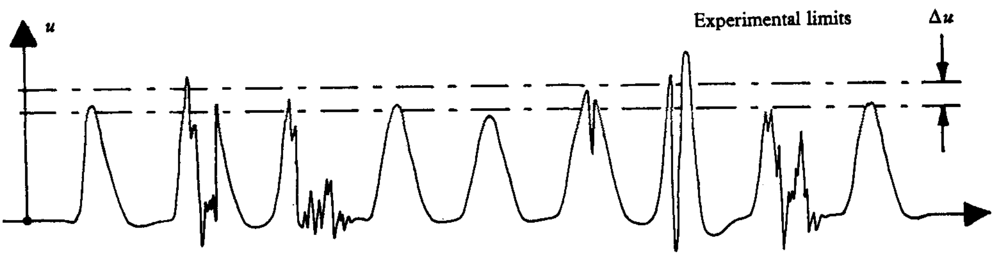

FIGURE 5. Oscillograph trace downstream of a regular sequence of spark pulses.

relay with a variable stop to the motion of the armature, so that the distance the cylinder penetrated the flow could be easily varied. Pulses of a few cycles/sec to operate the relay were obtained from a scale of 64 counter fed by a Beat Frequency Oscillator. This method was only moderately successful. The motion of the pulse was obscured by the presence of quasi-periodic oscillations and there was great variation in the pulse amplitude from pulse to pulse. The second method was that of a spark driven by discharging a condenser across the primary of a 1 : 100 transformer. Control over the pulse amplitude was obtained by varying the condensor charging voltage. It was found that even when no spark occurred that a velocity pulse was formed from the motion of the partially ionized air in the vicinity of the spark gap. No consideration was given to possible non-isothermal effects after a spark, since the ionization level was seen by visual observation with the laboratory blacked out to be negligible less than 1 cm (1 msec) downstream of the spark gap. Figure 5 shows a schematic oscillograph trace obtained from the hot-wire probe although in practice the signal was observed on a triggered oscilloscope. The voltage pulse was gradually increased until some of the pulses were turbulent. The critical velocity fluctuation u_m and the experimental limit Δu was determined such that (1) all pulses of amplitude greater than $u_m + \frac{1}{2}\Delta u$ had broken down, (2) no pulse of amplitude less than $u_m - \frac{1}{2}\Delta u$ had broken down. This procedure is complicated by the occasional extra-strong pulse for which turbulence is fully spread through the pulse, since the maximum values of u in the spot generally drop off rapidly as the eddies of smaller size appear in the spot. Successive pulses were sufficiently similar to allow u^* to be determined to $\pm 10\%$. Pulses were produced over the following ranges: $x = 15$–100 cm and $U_1 = 200$–1500 cm/sec, or $R = 2 \times 10^4$ to 10^6 and $X/\delta = 0{\cdot}5$–30.

It should be emphasized that the hot-wire probe was always downstream of the spark, the spark and the probe were separated by varying distances and that the critical pulse intensity was that which caused breakdown just ahead of the probe, rather than at the spark.

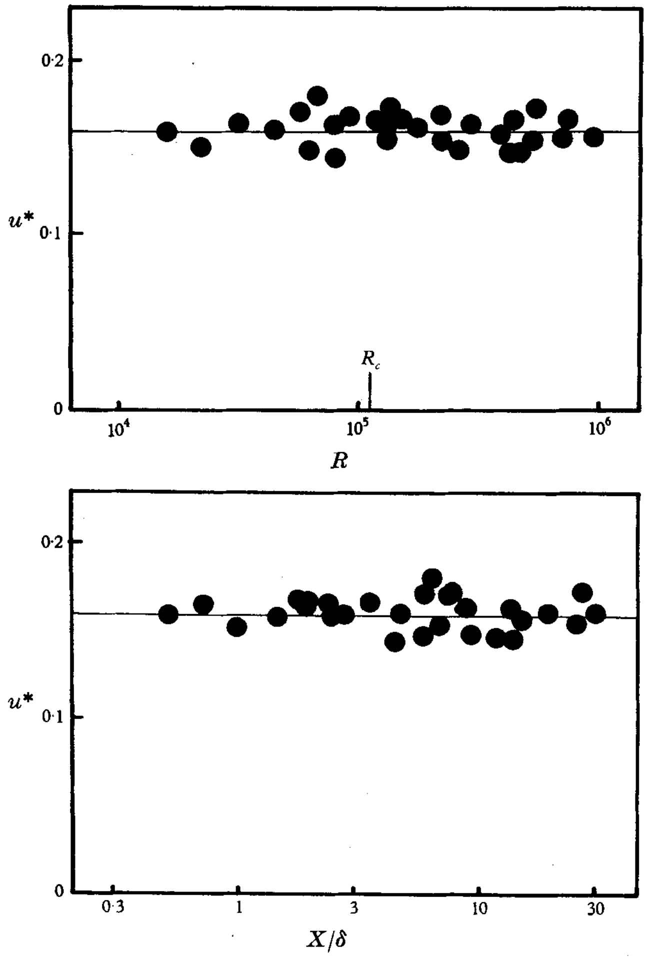

FIGURE 6. Critical intensity u^* at fixed $y/\delta = 0{\cdot}6$ as a function of $R = U_1 x/\nu$ and X/δ.

Figure 6 presents the main result of the paper u^* as a function of R and X/δ, obtained at $y/\delta = 0{\cdot}6$. It was not possible, however, to vary R and X/δ completely independently, but in the data of figure 6 for each value of R values of X/δ were possible covering a range of about 10:1. Thus it is simplest to regard figure 6 as presenting the same data in two different ways. This method is preferable to a three-dimensional diagram with axes u^*, R and X/δ, since R and X/δ are only slightly correlated.

The breakdown condition is independent of the Reynolds number over the 50:1 range 2×10^4 to 10^6. This remarkable result confirms the idea that breakdown is determined by local factors alone. It is also important to notice that breakdown is possible at Reynolds numbers much lower than $R_c = 1{\cdot}12 \times 10^5$, the critical value below which the layer is stable to all small disturbances (Schlichting 1955). It is also surprising that u^* is independent of X/δ. (The lateral extent of the pulses Z/δ was in the range 0·5–10 but was not controlled or generally measured.)

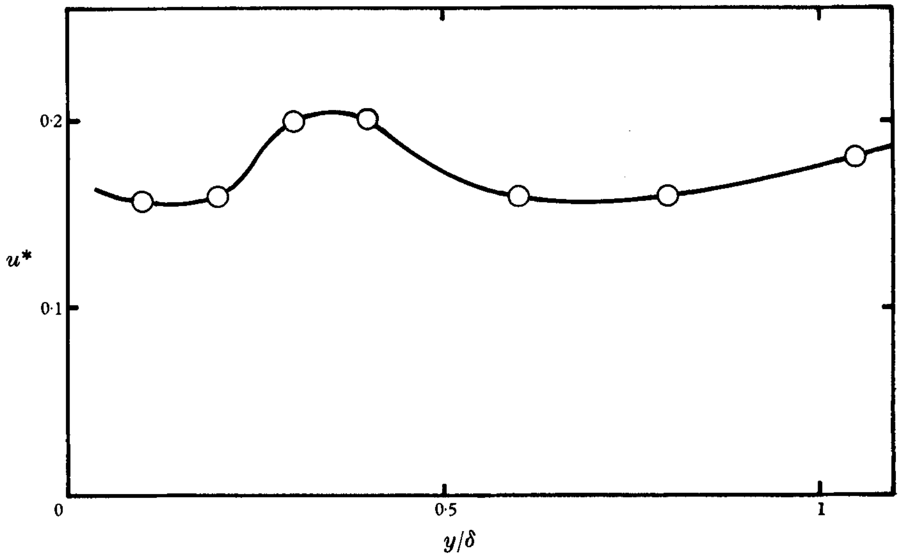

FIGURE 7. Critical intensity as a function of y/δ.

The critical intensity u^* is therefore a function only of y/δ and this relation is shown in figure 7. Over most of the layer $u^* \doteqdot 0{\cdot}18$.

It is now clear that transition to turbulence is largely independent of the particular way in which disturbances arise in a layer and that the Reynolds number which has hitherto been considered as an important parameter, directly affecting transition, is important merely in the processes responsible for the growth of the disturbances.

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